12/6/21

# Section 16.□: Divergence Theorem

Idea: Generalize Green's Theorem again, this time the version

$$\int_{\partial D} \vec{F} \cdot \vec{n}\, ds = \iint_D \operatorname{div}(\vec{F})\, dA$$

(look at Green's Theorem)

Prop (Divergence Theorem): Suppose $R$ is a solid region in $\mathbb{R}^3$ and $\vec{F}$ is a vf which has cts partials on $R$. If $R$ is a simple region, then $\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$.

NB: A simple region is in $\mathbb{R}^3$ is a solid with one boundary component (ie $\partial R$ is a single surface) which is piecewise smooth.

Non-ex:


Ex:

Solid Disk

Ex: Compute the flux of $\vec{F} = \langle x, y, z \rangle$ across $x^2+y^2+z^2=1$

Sol: Apply the divergence theorem:

$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}[x] + \frac{\partial}{\partial y}[y] + \frac{\partial}{\partial z}[z] = 3$

Noting $S = \partial R$ for $R$ the solid disk $x^2+y^2+z^2 \leq 1$

$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\partial R} \vec{F} \cdot ds = \iiint_R \operatorname{div}(\vec{F})\, dV = \iiint_R 3\, dV = 3\iiint 1\, dV = 3\operatorname{Vol}(R)$

$= 3\left(\frac{4}{3}\pi\, 1^3\right) = 4\pi$

(NB: Next we'll verify we get the same result from direct computation of the surface integral)

Sol 2: The surface is parameterized by

$S(\theta, \varphi) = \langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi)\rangle$
on $(\theta, \varphi) \in [0, 2\pi] \times [0, \pi]$

$\vec{S}_\theta = \langle -\sin(\varphi)\sin\theta, \sin(\varphi)\cos(\theta), 0\rangle$

$\vec{S}_\varphi = \langle \cos(\varphi)\cos(\theta), \cos(\varphi)\sin(\theta), -\sin(\varphi)\rangle$

$$\therefore \vec{S}_\theta \times \vec{S}_\varphi = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(\varphi)\sin(\theta) & \sin(\varphi)\cos(\theta) & 0 \\ \cos(\varphi)\cos(\theta) & \cos(\varphi)\sin(\theta) & -\sin(\varphi) \end{vmatrix}$$

$= \langle \sin^2(\varphi)\cos(\theta), -(-\sin^2(\varphi)\sin(\theta), -\sin(\varphi)\cos(\varphi)\sin^2(\theta) - \sin(\varphi)\cos(\varphi)\cos^2(\theta)\rangle$

$= -\sin(\varphi)\langle \sin(\varphi)\cos\theta, \sin(\varphi)\sin(\theta), \cos(\varphi)\rangle$

At $\theta = 0$, $\varphi = \frac{\pi}{2}$ we get $-\langle 1, 0, 0\rangle$ pointing inward

So we use $-\vec{S}_\theta \times \vec{S}_\varphi$ to correct orientation

$\therefore \vec{F}(\vec{S}(\theta,\varphi)) \cdot -(\vec{S}_\theta \times \vec{S}_\varphi) = \langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi)\rangle \cdot \sin(\varphi)\langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi)\rangle$

$= \sin(\varphi)(\sin^2(\varphi)\cos^2(\theta) + \sin^2(\varphi)\sin^2(\theta) + \cos^2(\varphi)) = \sin(\varphi)$

$\therefore \iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{S}(\theta,\varphi)) \cdot -(\vec{S}_\theta \times \vec{S}_\varphi)\, dA = \iint_D \sin(\varphi)\, dA$

$$= \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} \sin(\varphi)\,d\varphi\,d\theta = \int_{\theta=0}^{2\pi}\Big[-\cos(\varphi)\Big]_{\varphi=0}^{\pi} d\theta = \int_{\theta=0}^{2\pi}(1+1)\,d\theta = 2\int_{\theta=0}^{2\pi} d\theta$$

$$= 2\cdot 2\pi = 4\pi$$

NB: The two solutions gave the same answer, so we verify the divergence theorem (for this example)

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Ex: Calculate the flux of $\vec{F} = \langle xe^{y}, z-e^{y}, -xy\rangle$ across the ellipsoid $x^2+2y^2+3z^2=4$

Solution: Apply the Divergence theorem:

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\,dV$$

for $R$ the solid ellipsoid $x^2+2y^2+3z^2=4$ (b/c $\partial R = S$)

but $\operatorname{div}(\vec{F}) = \nabla\cdot\vec{F} = e^{y} + (-e^{y}) + 0 = 0$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = \iiint 0\,dV = 0$$

NB: We could verify this directly via a parameterization of $S$. The parameterization is indicated by

$$x^2+2y^2+3z^2=4$$

iff $\dfrac{x^2}{6}+\dfrac{y^2}{3}+\dfrac{z^2}{2} = \dfrac{4}{6} = \dfrac{2}{3}$

iff $\left(\dfrac{x}{\sqrt{6}}\right)^2+\left(\dfrac{y}{\sqrt{3}}\right)^2+\left(\dfrac{z}{\sqrt{2}}\right)^2 = \left(\sqrt{\dfrac{2}{3}}\right)^2$

We can parameterize the solid ellipsoid using a version of spherical coords

$$\begin{cases} x = \sqrt{6}\,\rho \sin(\varphi)\cos(\theta) \\ y = \sqrt{3}\,\rho \sin(\varphi)\sin(\theta) \\ z = \sqrt{2}\,\cos(\varphi) \end{cases}$$

↳ yield $\rho^2 = \frac{2}{3}$ is the surface

$\left(\text{Check } \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \rho^2 \sin(\varphi)\right)$

Ex: Compute flux of $\vec{F} = \langle 3x, xy, 2xz \rangle$ across $\partial[0,1]^3$

NB: Parameterizing this surface would require 6 different pieces...
But w/ divergence theorem, I might not have to

Solution: Applying the divergence theorem:

$$\iint_{\partial[0,1]^3} \vec{F} \cdot d\vec{S} = \iiint_{[0,1]^3} \operatorname{div}(\vec{F})\, dV$$

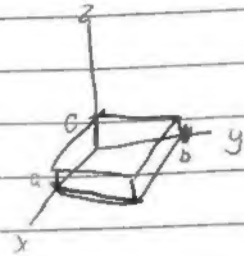
$$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}[3x] + \frac{\partial}{\partial y}[xy] + \frac{\partial}{\partial z}[2xz] = 3 + x + 2x = 3 + 3x$$

$$\therefore \iint_{\partial[0,1]^3} \vec{F} \cdot d\vec{S} = \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} 3 + 3x \; dx\,dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1} \left[3x + \frac{3}{2}x^2\right]_{x=0}^{1} dy\,dz = \int_{z=0}^{1}\int_{y=0}^{1} 3 + \frac{3}{2} - 0 \; dy\,dz = 3\left(\frac{3}{2}\right)[0,1]^2$$

$$= \frac{9}{2}(1-0)(1-0) = \frac{9}{2}$$

Ex: Calculate the flux of $\vec{F} = \langle x^2yz, xy^2z, xyz^2 \rangle$ across the boundary of the rectangular box $[0,a] \times [0,b] \times [0,c]$ for $a,b,c>0$




Solution: Let's apply the divergence theorem:

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_{R} \operatorname{div}(\vec{F})\, dV$$

$$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = 2xyz + 2xyz + 2xyz = 6xyz$$

$$\therefore \iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_{R} 6xyz\, dV$$

$$= \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} 6xyz\, dz\, dy\, dx = \int_{x=0}^{a} \int_{y=0}^{b} 3\left[xyz^2\right]_{z=0}^{c} dy\, dx$$

$$= \int_{b=0}^{a} \int_{y=0}^{b} 3(xyc^2 - 0)\, dy dx = 3c^2 \int_{x=0}^{a} \int_{y=0}^{b} xy\, dy\, dx = 3c^2 \int_{b=0}^{a} \left[\tfrac{1}{2} x y^2\right]_{y=0}^{b} dx$$

$$= 3c^2 \int_{x=0}^{a} \tfrac{1}{2} x\, b^2 - 0\, dx = \tfrac{3}{2} c^2 b^2 \int_{x=0}^{a} x\, dx = \tfrac{3}{4} b^2 c^2 (a^2 - 0) = \tfrac{3}{4} a^2 b^2 c^2$$